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Proof of a conjecture of W. Veys

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1. INTRODUCTION

In this thesis [V], W. Veys formulates the following

Conjecture. *Let C_i, $i = 1, \dots, n$, be distinct irreducible algebraic curves in the complex projective plane $\boldsymbol{P}^2(\mathbb{C})$. If the topological Euler characteristic $e(\boldsymbol{P}^2(\mathbb{C}) \setminus \bigcup_{i=1}^n C_i) \leq 0$, then each C_i is a rational curve.*

In this note we prove this conjecture. The main argument consists of the following inequality between Euler characteristics.

Proposition 1. *Let $(X_t)_{t \in L}$ be a pencil of plane curves. Suppose that the general member F of this pencil is irreducible. Then $e(X_t) \geq e(F)$ for all $t \in T$.*

The following example of W. Veys shows, that the hypothesis in the conjecture cannot be weakened:
Let $C_1 = V(x^3 + z^3 - y^2 z)$, $C_2 = V(x^3 - y^2 z)$ and $C_3 = V(z)$. Then C_1 has genus one and $e(\boldsymbol{P}^2(\mathbb{C}) \setminus \bigcup_{i=1}^3 C_i) = 1$.

In § 2 we study the vanishing cycle sheaves of one-parameter families of curves, and prove some general facts which imply Proposition 1. Finally, in § 3 we complete the proof of Veys' conjecture.

In this section X denotes a reduced complex surface, S denotes the unit disc in $\mathbb{C}$ and $f : X \to S$ is a proper flat holomorphic mapping, such that the mapping $X \backslash X_0 \to S \backslash \{0\}$ induced by f is a topological fibre bundle. Here for $t \in S$ we let $X_t = f^{-1}(t)$.

For a bounded constructible complex K on X we let $\psi_f(K)$ denote the complex of nearby cycles of K for f. If $\mathfrak{H}$ denotes the complex upper half plane and $e : \mathfrak{H} \to S \backslash \{0\}$ is given by $e(u) = \exp(2\pi i u)$, we let $X_\infty = X \times_S \mathfrak{H}$ and let $k : X_\infty \to X$, $i : X_0 \to X$ denote the natural mappings. Then $\psi_f(K) := i^* R k_* k^*(K)$. The natural morphism $i^*(K) \to \psi_f(K)$ fits in the distinguished triangle

$$i^*(K) \longrightarrow \psi_f(K) \longrightarrow \phi_f(K) \xrightarrow{+1}$$

and induces a long exact cohomology sequence (the *specialization sequence*)

$$\ldots \to \boldsymbol{H}^j(X_0, i^*K) \to \boldsymbol{H}^j(X_0, \psi_f(K)) \to \boldsymbol{H}^j(X_0, \phi_f(K)) \to \boldsymbol{H}^{j+1}(X_0, i^*K) \to \ldots$$

Let us consider the case $K = \mathbb{C}_X$. Then $\boldsymbol{H}^j(X_0, \psi_f(K)) \simeq H^j(X_\infty, \mathbb{C})$ so in the previous exact sequence the cohomology of the general and special fibres of f can be compared.

If Y is an n-dimensional complex space, a perverse sheaf on Y is a bounded complex K of $\mathbb{C}_Y$-modules, such that

$$\begin{cases} \dim \operatorname{supp} \mathcal{H}^{-i}K \le i & \forall i \in \mathbb{Z}; \\ \dim \operatorname{supp} \mathcal{H}^{-i}DK \le i & \forall i \in \mathbb{Z}. \end{cases}$$

Here DK denotes the Verdier dual of K [BBD, p. 102]. In particular, for Y smooth one has $\mathcal{H}^{-i}\mathbb{C}_Y[n] = 0$ for $i \neq n$, $\mathcal{H}^{-n}\mathbb{C}_Y[n] = \mathbb{C}_Y$ and $D(\mathbb{C}_Y[n]) = \mathbb{C}_Y[n]$, hence $\mathbb{C}_Y[n]$ is perverse. In the singular case, $D(\mathbb{C}_Y[n]) = D_Y^{\cdot}[-n]$ with $D_Y^{\cdot}$ the dualizing complex on Y. For $y \in Y$, $\mathcal{H}^{-i}(D_Y^{\cdot})_y \cong \operatorname{Hom}(H^i_{\{y\}}(\mathbb{C}_Y), \mathbb{C})$. It follows from [H, Lemma 4] that $\operatorname{supp} \mathcal{H}^{-n-i}(D_Y^{\cdot})$ has dimension $\le i$ if Y is locally a complete intersection, hence $\mathbb{C}_Y[n]$ is perverse in that case. If K is perverse and $f : Y \to \mathbb{C}$, then $\psi_f(K)[-1]$ and $\phi_f(K)[-1]$ are perverse ([B, Proposition 2.3.6 and 2.3.9]).

Lemma 1. *Suppose that the sequence of cohomology sheaves*

$$0 \to \mathbb{C}_{X_0} \to \mathcal{H}^0(\psi_f(\mathbb{C})) \to \mathcal{H}^0(\phi_f(\mathbb{C})) \to 0$$

is split exact. Then the mapping $\operatorname{sp} : H^1(X_0, \mathbb{C}) \to H^1(X_\infty, \mathbb{C})$ *from the specialization sequence is injective.*

Proof. Observe that $\boldsymbol{H}^0(X_0, \psi_f(K)) \simeq H^0(X_0, \mathcal{H}^0(\psi_f(K)))$ and similarly for ϕ_f instead of ψ_f. As the cohomology sheaf sequence is split exact, we conclude that the natural map

$$H^0(X_0, \mathcal{H}^0(\psi_f(K))) \to H^0(X_0, \mathcal{H}^0(\phi_f(K)))$$

is surjective. $\square$

Example. Let X be the union of two planes in $\boldsymbol{P}^4(\mathbb{C})$ which intersect in one point. Consider the pencil of curves obtained by intersecting X with a general pencil of hyperplanes. The general fibre will be the disjoint union of two lines, where one fibre X_0 is the union of two intersecting lines. Let $\tilde{X}_0 \xrightarrow{n} X_0$ denote the normalization. Then the sequence of Lemma 1 reads

$$0 \to \mathbb{C}_{X_0} \to n_*\mathbb{C}_{\tilde{X}_0} \to \mathbb{C}_P \to 0$$

where P is the point of intersection of the two planes. This sequence is not split.

Lemma 2. *Suppose that X is a locally complete intersection and that $X^{\text{reg}} \cap X_0$ is dense in X_0. Then the condition of Lemma 1 is fulfilled.*

Proof. As we noticed above, the sheaf complex $\mathbb{C}_X[2]$ is perverse. Hence $\psi_f(\mathbb{C}_X)[1]$ and $\phi_f(\mathbb{C}_X)[1]$ are perverse. The monodromy operator T acts on these perverse sheaves, and we have a decomposition

$$\psi_f(\mathbb{C}_X) = \psi_f(\mathbb{C}_X)^1 \oplus \psi_f(\mathbb{C}_X)^{\neq 1}$$

into perverse subsheaves on which $T - \mathrm{Id}$ is nilpotent (resp. an isomorphism). This follows because the category of perverse sheaves on X_0 is abelian. We will show that under the hypothesis of the lemma, $\mathbb{C}_{X_0}$ maps isomorphically to $\mathcal{H}^0(\psi_f(\mathbb{C}_X)^1)$. The geometric meaning of this statement is, that for $x \in X_0$ with Milnor fibre $X_{f,x}$, T acts on $H^0(X_{f,x})$ with a single eigenvalue 1.

At any point x of X_0 where X_0^{red} and X are smooth, f is given in local coordinates by $(z, w) \mapsto z^e$ for some $e \in \mathbb{N}$. In that case, $X_{f,x}$ consists of e connected components which are permuted cyclically by T, so the statement is true in these points. As these form a dense Zariski-open subset of X_0, we only have to deal with a finite set of points. Suppose that $x \in X_0$ and $\mathbb{C}_{X_0,x} \to \mathcal{H}^0(\psi_f(\mathbb{C}_X)^1_x)$ is not surjective. Then $\mathcal{H}^0(\phi_f(\mathbb{C}_X))$ and hence $\phi_f(\mathbb{C}_X)$ would have a section with support in x; this contradicts the perversity of $\phi_f(\mathbb{C}_X)$. We conclude that $\mathbb{C}_{X_0,x} \to \mathcal{H}^0(\psi_f(\mathbb{C}_X)^1_x)$ is surjective. $\square$

Remark. In the example above, the sheaf $\mathbb{C}_X$ is not perverse; the reason is that $H^1_{\{P\}}(\mathbb{C}_X) \neq 0$.

Proof of Proposition 1. Let $b_i(M) = \dim_{\mathbb{C}} H^i(M, \mathbb{C})$ for a space M. Then $e(X_t) = b_0(X_t) - b_1(X_t) + b_2(X_t)$, $e(F) = b_0(F) - b_1(F) + b_2(F)$. Every plane curve is connected, so $b_0(X_t) = b_0(F) = 1$. As F is irreducible, $b_2(F) = 1 \leq b_2(X_t)$, and the pencil has only isolated fixed points. Hence we may apply Lemmas 1 and 2 to $X = \{(x, t) \in \mathbb{P}^2 \times L \mid x \in X_t\} \to L$ and conclude that $b_1(X_t) \leq b_1(F)$. Hence $e(X_t) \geq e(F)$. $\square$

3. PROOF OF THE CONJECTURE

We proceed by induction on the number n of curves. The case $n = 1$ is easy: an irreducible curve C has $e(C) \leq 2$ so $e(\boldsymbol{P}^2(\mathbb{C}) \setminus C) \geq 1$. Similarly, if C_1 and C_2 are irreducible then $e(C_1 \cup C_2) \leq 3$ with equality iff C_1 and C_2 are homeomorphic to S^2 and intersect in precisely one point. So in the sequel we suppose that $n \geq 3$. We let $U = \boldsymbol{P}^2(\mathbb{C}) \setminus \bigcup_{i=1}^n C_i$ and put $C_i^0 = C_i \setminus \bigcup_{j \neq i} C_j$. If $e(C_i^0) \leq 0$ we can apply the induction hypothesis to $U' = U \cup C_i^0$ and conclude that C_j is rational for $j \neq i$. Moreover, we remark that $e(C_i^0) > 0$ implies that C_i is rational. So a minimal counterexample to the conjecture would have the following properties (after renumbering):

- C_1 not a rational curve;
- $e(C_i^0) = 1$ for $i \geq 2$.

We will derive a contradiction in this situation. Observe that the last property means that there exists $P \in \boldsymbol{P}^2(\mathbb{C})$ so that $C_i \cap C_j = \{P\}$ for all $i \neq j$, and that the normalization of C_i is homeomorphic to C_i for all $i \geq 2$.

Let F_i be an irreducible homogeneous polynomial defining the curve C_i and let $d_i = \deg(F_i)$. Let $lcm(d_1, d_2) = d = n_1 d_1 = n_2 d_2$ and consider the pencil of curves of degree d spanned by $n_1 C_1$ and $n_2 C_2$:

$$X = \{((x_0 : x_1 : x_2), (\lambda_1 : \lambda_2)) \in \boldsymbol{P}^2(\mathbb{C}) \times \boldsymbol{P}^1(\mathbb{C}) \mid \lambda_1 F_1(x_0, x_1, x_2)^{n_1} + \lambda_2 F_2(x_0, x_1, x_2)^{n_2} = 0\}$$

and let $f : X \to \boldsymbol{P}^1(\mathbb{C})$ denote the projection on the second factor. Observe that the map $X \to \boldsymbol{P}^2(\mathbb{C})$ is the blowing up of the zero-dimensional subscheme defined by $(F_1^{n_1}, F_2^{n_2})$ with exceptional divisor $E = \{P\} \times \boldsymbol{P}^1(\mathbb{C})$; hence $X \setminus E$ is smooth.

Lemma 3. *The general fibre of f is irreducible.*

Proof. Let $\nu : X' \to X$ denote the normalization of X and $X' \xrightarrow{g} C \xrightarrow{\pi} \boldsymbol{P}^1(\mathbb{C})$ the Stein factorization of $f\nu$. As X is rational, C is again a rational curve. Recall that X is nonsingular outside E, so the restriction of ν to each fibre of f is birational. We conclude that $(f\nu)^{-1}(0) = \nu^{-1}(C_1)$ and $(f\nu)^{-1}(\infty) = \nu^{-1}(C_2)$ are irreducible. Moreover $g(\nu^{-1}(C_1))$ and $g(\nu^{-1}(C_2))$ are distinct points of C, say 0 and ∞. The morphism π ramifies totally in 0 and ∞, because the fibres of $f\nu$ over 0 and ∞ are irreducible. Hence the multiplicity of $\nu^{-1}(C_i)$ as a fibre of $f\nu = \pi g$ is divisible by $\deg(\pi)$ for $i = 1, 2$. Hence $\deg(\pi)$ divides n_1 and n_2, so $\deg(\pi) = 1$. □

Lemma 4. *f is constant on each C_i.*

Proof. The pencil of curves spanned by $n_1 C_1$ and $n_2 C_2$ does not move on the normalization C_i' of C_i. □

We let $\{\alpha_1, \ldots, \alpha_k\}$ denote the set of images of $C_3, \ldots, C_n$ under f, and define:

$$C(\alpha_j) = \bigcup_{f(C_i) = \alpha_j} C_i, \qquad B_j^0 = f^{-1}(\alpha_j) \setminus C(\alpha_j);$$

$$U'' = f^{-1}(\boldsymbol{P}^1(\mathbb{C})\backslash\{0,\infty,\alpha_1,\ldots,\alpha_k\}), \qquad U' = U \cap U''$$

where we consider U as a subset of X. Then $U = U' \coprod \bigcup_{j=1}^{k} B_j^0$. Let X_η denote a general fibre of f. Then one has:

$$(1) \qquad \begin{cases} 0 \geq e(U) \\ \quad = e(U') + \sum_j e(B_j^0) \\ \quad = e(U'') - e(E\backslash f^{-1}(\{0,\infty,\alpha_1,\ldots,\alpha_k\}) + \sum_j e(B_j^0) \\ \quad = e(X_\eta)e(\boldsymbol{P}^1(\mathbb{C})\backslash\{0,\infty,\alpha_1,\ldots,\alpha_k\}) \\ \qquad + \sum_{s\in f(U'')} (e(X_s) - e(X_\eta)) + k + \sum_j e(B_j^0) \\ \quad \geq k(1-e(X_\eta)) + \sum_j e(B_j^0). \end{cases}$$

For each j there are two possibilities:

Case 1: $B_j^0 = \emptyset$; then $e(B_j^0) = 0$.
Case 2: $B_j^0 \neq \emptyset$; then we have

$$e(B_j^0) = e(X_{\alpha_j}) - e(C(\alpha_j)) = [e(X_{\alpha_j}) - e(X_\eta)] + e(X_\eta) - e(C(\alpha_j)).$$

Now put $r_j = \#\{i \mid C_i \subseteq X_{\alpha_j}\}$. As $B_j^0 \neq \emptyset$, X_{α_j} has at least $r_j + 1$ components, so $e(X_{\alpha_j}) \geq e(X_\eta) + r_j$, in other words

$$e(B_j^0) \geq r_j + e(X_\eta) - (r_j+1) = e(X_\eta) - 1.$$

From (1) we now obtain

$$(2) \qquad 0 \geq (\#\{j \mid B_j^0 \neq \emptyset\} - k)(e(X_\eta) - 1).$$

This implies that either $e(X_\eta) \geq 1$ (so the general fibre has genus 0, so C_1 has genus 0 too: a contradiction) or $B_j^0 \neq \emptyset$ for $j = 1,\ldots,k$ and equality holds everywhere in (1). So

$$e(X_{\alpha_j}) = e(B_j^0) + e(C(\alpha_j)) = e(X_\eta) - 1 + r_j + 1 = e(X_\eta) + r_j.$$

Suppose that $e(X_\eta) \leq 0$. Since $b_2(X_{\alpha_j}) \geq r_j + 1$,

$$b_1(X_{\alpha_j}) \geq -e(X_{\alpha_j}) + 1 + r_j + 1 = b_1(X_\eta) \geq b_1(X_{\alpha_j})$$

by Lemmas 1 and 2, so $b_1(X_{\alpha_j}) = b_1(X_\eta)$ for $j = 1,\ldots,k$. This and the fact that equality holds in (1) implies that for each critical value t of f with $t \neq 0, \infty$ the specialization map $H^1(X_t,\mathbb{C}) \to H^1(X_\eta,\mathbb{C})$ is an isomorphism. This means that the restriction of $R^1 f_* \mathbb{C}_X$ to $\boldsymbol{P}^1(\mathbb{C})\backslash\{0,\infty\}$ is locally constant, and has a cyclic monodromy group.

Let $\nu : X' \to X$ again denote the normalization of X and $f' = f\nu$. The general fibre X'_η of f' is the normalization of X_η and we conclude that the restriction of $R^1 f'_* \mathbb{C}_{X'}$ to the set of regular values of f' is a polarized variation of Hodge structure with cyclic monodromy group. As the monodromy representation is also completely reducible, we conclude that the monodromy around 0 and ∞ is of finite order, say e, and the period mapping associated to f' is constant. Let $g : Y \to \boldsymbol{P}^1(\mathbb{C})$ be a minimal model for f': i.e. Y is smooth projective and contains no (-1)-curves contained in a fibre. Observe that Y is still a rational surface, so $b_1(Y) = 0$. Let V be the set of regular values of g and let $Y^0 = g^{-1}(V)$, $g^0 : Y^0 \to V$. The local system $R^1 g^0_* \mathbb{Q}_{Y^0}$ has a finite cyclic monodromy group generated by the local monodromy transformation around 0. Moreover, the normalization of C_1 occurs as an irreducible component of $g^{-1}(0)$, because $g(C_1) > 0$, hence the strict transform of C_1 is not an exceptional curve of first kind on any desingularization of X'. By Proposition 1, the specialization map $H^1(Y_0, \mathbb{Q}) \to H^1(Y_\eta, \mathbb{Q})$ is injective and its image is invariant under the monodromy group. Moreover $H^1(Y_0, \mathbb{Q}) \neq 0$ as C_1 is not rational. We obtain that $H^0(V, R^1 g^0_* \mathbb{Q}_{Y^0}) \neq 0$. By the invariant cycle theorem ([D] Theorem 4.1.1(ii)) we have a surjection $H^1(Y, \mathbb{Q}) \to H^0(V, R^1 g^0_* \mathbb{Q}_{Y^1})$. Hence $b_1(Y) > 0$: a contradiction. $\square$

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